The Chorin-Temam method and the BDF-2 method for the incompressible Navier-Stokes equations

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1 General setting and incompressible Navier-Stokes system

Consider the a regular domain $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ and the incompressible Navier-Stokes system:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in} \quad \Omega \\ \mathbf{u} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

In this case, $\mathbf{u} \in \mathbb{R}^d$ is the velocity field, $p \in \mathbb{R}$ is the pressure field, $\mathbf{f} \in \mathbb{R}^d$ are the external forces (gravity for example) and we enforce homogeneous Dirichlet boundary conditions for the sake of simplicity.

Remark 1.1 (Boussinesq's approximation)**.** *We have done the Boussinesq's approximation, so that we have divided all the terms of the first equation by ρ_0 and we have takend the normalized pressure p to actually be p/ρ_0.*

Remark 1.2 (The role of the pressure p)**.** *It is important to notice that now the pressure p does not have any thermodynamical interpretation as it is the case for the compressible Navier-Stokes system. In the latter case, the mass balance (second equation) and the linear momentum balance (first equation) are not sufficient to close the system, and then we need an energy balance (not listed here). In the incompressible case, the energy balance is decoupled from the mass and the momentum balance. Moreover, the pressure p becomes nothing but a Lagrange multiplier enforcing the constraint:*

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

Remark 1.3 (The role of the pressure p bis)**.** *Consider Eq.* (1) *where we neglect the inertial term $\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$ (Stokes system). Then, under an appropriate choice of search spaces $\mathcal{V}$ for the velocity and $\mathcal{Q}$ for the pressure, setting:*

$$\mathcal{L}(\mathbf{v}, q) = \int_\Omega \frac{\nu}{2} |\nabla \mathbf{v}|^2 d\mathbf{x} - \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} - \int_\Omega (\nabla \cdot v)\, q d\mathbf{x}, \tag{3}$$

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we can show the equivalence between problem (1) *and the following saddle-point problem:*

$$\mathcal{L}(\mathbf{u}, p) = \max_{q \in \mathcal{Q}} \min_{\mathbf{v} \in \mathcal{V}} \mathcal{L}(\mathbf{v}, q). \tag{4}$$

This shows the role of p as Lagrange multiplier for the incompressibility constraint.

Remark 1.4 (A piece of folklore on the Navier-Stokes system)**.** *Though the Navier-Stokes system* (1) *may seem simple, the existence and the regularity of the solution for $d = 3$ is one of the "problems of the millenary" of the Clay institute. The proof for the case $d = 2$ has been done in 1958 by Ladyzhenskaya [4] and relies on the following form of the more general Gagliardo-Niremberg inequality: if u (scalar function of two variables) is a weakly differentiable function vanishing on $\partial\Omega$ in the sense of the trace, there exists $C = C(\Omega) > 0$ such that:*

$$\|u\|_{L^4(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{[L^2(\Omega)]^2}^{1/2}. \tag{5}$$

In the case $d = 3$, the inequality becomes

$$\|u\|_{L^4(\Omega)} \leq C \|u\|_{L^2(\Omega)}^{1/4} \|\nabla u\|_{[L^2(\Omega)]^3}^{3/4}, \tag{6}$$

which is not enough to conclude.

Remark 1.5 (2D and 3D are not the same)**.** *Consider the Navier-Stokes system for $d = 2$ or for $d = 3$ has a huge impact on the outcome, especially as far as turbulence is concerned. Consider the vorticity $\boldsymbol{\omega}$, defined as:*

$$\boldsymbol{\omega} = \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right) \mathbf{e}_z, \tag{7}$$

for $d = 2$ and by:

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{vmatrix}, \tag{8}$$

for $d = 3$. If we compute the Navier-Stokes equation for the vorticity (take the curl of momentum balance of the Navier-Stokes system and use some vectorial identity), we gain:

$$\frac{D\boldsymbol{\omega}}{Dt} := \partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \nu \Delta \boldsymbol{\omega} + \nabla \times \mathbf{f}, \tag{9}$$

for $d = 2$ and

$$\frac{D\boldsymbol{\omega}}{Dt} := \partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \underbrace{\boldsymbol{\omega} \cdot \nabla \mathbf{u}}_{!!!} + \nu \Delta \boldsymbol{\omega} + \nabla \times \mathbf{f}, \tag{10}$$

for $d = 3$. What makes 3D turbulence way different from 2D turbulence is the presence of the term $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$, which vanishes in the two dimensional case since the vorticity is orthogonal to the plan of motion.

2 The Chorin-Temam method

An intuitive way of discretizing the Navier-Stokes system in time would be, once we consider a time-step Δt, to consider:

$$\begin{cases} \frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\Delta t} + \mathbf{u}^\star \cdot \nabla \mathbf{u}^{\star\star} - \nu \Delta \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u}^{n+1} = 0 & \text{in} \quad \Omega \\ \mathbf{u}^{n+1} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{11}$$

where $\mathbf{u}^\star$ and $\mathbf{u}^{\star\star}$ can be either $\mathbf{u}^n$ or $\mathbf{u}^{n+1}$, giving birth to different kind of difficulties, stability constraints, etc. This problem is rather hard to treat due to the coupling between the velocity and the pressure field, so we decide to perform an operator splitting and solve a first problem with solution $\tilde{\mathbf{u}}^{n+1}$, for which in general

$$\nabla \cdot \tilde{\mathbf{u}}^{n+1} \neq 0 \qquad \text{on} \quad \Omega, \tag{12}$$

called "prediction step", followed by the solution of another problem (called "correction step") which provides $\mathbf{u}^{n+1}$ satisfying the incompressibility constraint. This is:

$$(\mathbb{P}) \quad : \quad \begin{cases} \frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\Delta t} + \mathbf{u}^\star \cdot \nabla \mathbf{u}^{\star\star} - \nu \Delta \tilde{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1} & \text{in} \quad \Omega \\ \mathrm{BC}_{\mathbb{P}}. \end{cases} \tag{13}$$

and

$$(\mathbb{C}) \quad : \quad \begin{cases} \frac{\mathbf{u}^{n+1}-\tilde{\mathbf{u}}^{n+1}}{\Delta t} + \nabla p^{n+1} = 0 & \text{in} \quad \Omega \\ \nabla \cdot \mathbf{u}^{n+1} = 0 & \text{in} \quad \Omega \\ \mathrm{BC}_{\mathbb{C}}, \end{cases} \tag{14}$$

where the boundary conditions are not specified for the moment. Nevertheless, we still have a coupled equation between an unknown velocity and and unknown pressure, thus we have to work a little bit harder.

We formally take the divengence of the first equation out of (14), which yields:

$$\underbrace{\frac{\nabla \cdot \mathbf{u}^{n+1}}{\Delta t}}_{=0} - \frac{\nabla \cdot \tilde{\mathbf{u}}^{n+1}}{\Delta t} + \nabla \cdot (\nabla p^{n+1}) = 0, \tag{15}$$

where we have used the second equation of (14). Remembering that $\nabla \cdot (\nabla q) = \Delta q$, we obtain a new system for the pressure:

$$(\mathbb{C}) \quad : \quad \begin{cases} -\Delta p^{n+1} = -\frac{\nabla \cdot \tilde{\mathbf{u}}^{n+1}}{\Delta t} & \text{in} \quad \Omega \\ \mathrm{BC}_{\mathbb{C}}, \end{cases} \tag{16}$$

which is nothing but a Poisson's problem we can solve using our favourite solver. One may ask about the destiny of $\mathbf{u}^{n+1}$: taking again the first equation of (14), we obtain the so-called "update step":

$$(\mathbb{U}) \quad : \quad \mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \Delta t \nabla p^{n+1}. \tag{17}$$

Modulo some technicalities about the boundary conditions (ask me if you want to know), we have recovered the so-called "Chorin-Temam" method (introduced in 1967-68 by Chorin [1]), which is resumed as follows:

1. **Prediction step**: Solve the advection-diffusion equation for the predicted velocity field $\tilde{\mathbf{u}}^{n+1}$ fulfilling the problem:

$$(\mathbb{P}) \quad : \quad \begin{cases} \frac{\tilde{\mathbf{u}}^{n+1}-\mathbf{u}^n}{\Delta t} + \mathbf{u}^\star \cdot \nabla \mathbf{u}^{\star\star} - \nu \Delta \tilde{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1} & \text{in} \quad \Omega \\ \tilde{\mathbf{u}}^{n+1} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{18}$$

   This can be done using a numerical method such Finite Volumes, Finite Differences (look at [5], which also presents the Chorin-Temam method) or Finite Elements (the bible is [2], something more friendly [3]).

2. **Correction step**: Solve the following Poisson problem for the pressure variable p^{n+1}:

$$(\mathbb{C}) \quad : \quad \begin{cases} -\Delta p^{n+1} = -\frac{\nabla \cdot \tilde{\mathbf{u}}^{n+1}}{\Delta t} & \text{in} \quad \Omega \\ \nabla p^{n+1} \cdot \mathbf{n} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{19}$$

   where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$. Again, this can be solved by our favourite solver.

3. **Projection step** (of $\tilde{\mathbf{u}}^{n+1}$ onto the divergence-free space): compute

$$(\mathbb{U}) \quad : \quad \mathbf{u}^{n+1} = \tilde{\mathbf{u}}^{n+1} - \Delta t \nabla p^{n+1} \qquad \text{on} \quad \Omega. \tag{20}$$

   This is often done on the weak form of this equation.

Remark 2.1 (Splitting error and boundary layer)**.** *After some computations done in the details we have skipped, one can show that only the condition:*

$$\mathbf{u}^{n+1} \cdot \mathbf{n} = 0 \qquad \textit{on} \quad \partial\Omega, \tag{21}$$

is actually enforced on the velocity field through the Chorin-Temam method. This creates a boundary layer on the pressure field with exponential decay, whose thickness h_{BL} has been estimated to be:

$$h_{BL} \propto \sqrt{\nu \Delta t}. \tag{22}$$

This is one of the contributions to the overall splitting error of the method.

Remark 2.2 (Pressure problem not well posed)**.** *The value of the pressure in Eq.* (19) *is defined up to a constant, since the boundary conditions are not enough to fix it. This is not due to the Chorin-Temam method, since it was already the case for the full Navier-Stokes system* (1)*. In practice, one fixes the value of the pressure at some point in the computational mesh to an arbitrary value, or enforces that the pressure field has zero average on the domain.*

Remark 2.3 (Stationary solution)**.** *Another drawback of the Chorin-Temam method is that (try to prove it) in the case of stationary solution, the obtained solution is not consistent with the stationary solution of the full Navier-Stokes problem.*

3 The BDF-2 (Backward Differentiation Formula) method

The Backward Differentiation Formula of order 2 is nothing but a method to increase the order of the time discretization of the Navier-Stokes system (1). Of course, it can be coupled with the Chorin-Temam method we have seen so far without major difficulties.

We first illustrate the idea behind this method on a simple example: consider the ordinary differential equation

$$\begin{cases} d_t\psi(t) & = \Phi\left(\psi(t), t\right) \\ \psi(t=0) & = \psi_0, \end{cases} \tag{23}$$

for some smooth function Φ which can depend also directly on time. We decide to discretize the time derivative in the following way:

$$d_t\psi(t^{n+1}) \simeq \alpha\psi(t^{n+1}) + \beta\psi(t^n) + \gamma\psi(t^{n-1}). \tag{24}$$

Supposing ψ smooth enough and performing Taylor expansions around t^{n+1} yields:

$$d_t\psi(t^{n+1}) = \left(\alpha+\beta+\gamma\right)\psi(t^{n+1}) + \Delta t\left(-\beta-2\gamma\right)d_t\psi(t^{n+1}) \tag{25}$$

$$+\,\Delta t^2\left(\frac{\beta}{2}+2\gamma\right)d_{tt}\psi(t^{n+1}) + O(\Delta t^3). \tag{26}$$

Enforcing consistency up to order 2:

$$\begin{cases} \alpha+\beta+\gamma = 0 \\ \Delta t\left(-\beta-2\gamma\right) = 1 \\ \Delta t^2\left(\frac{\beta}{2}+2\gamma\right) = 0 \end{cases} \quad \to \quad \begin{cases} \alpha = \frac{3}{2\Delta t} \\ \beta = -\frac{2}{\Delta t} \\ \gamma = \frac{1}{2\Delta t}. \end{cases} \tag{27}$$

This results in the BDF-2 formula:

$$d_t\psi(t^{n+1}) \simeq \frac{1}{\Delta t}\left(\frac{3}{2}\psi^{n+1} - 2\psi^n + \frac{1}{2}\psi^{n-1}\right). \tag{28}$$

Applying this idea to the Navier-Stokes system, we obtain:

$$\begin{cases} \frac{3\mathbf{u}^{n+1}-4\mathbf{u}^n+\mathbf{u}^{n-1}}{2\Delta t} + \mathbf{u}^\star\cdot\nabla\mathbf{u}^{\star\star} - \nu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} & \text{in } \Omega \\ \nabla\cdot\mathbf{u}^{n+1} = 0 & \text{in } \Omega \\ \mathbf{u}^{n+1} = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

We have to be slightly more careful about the choice of $\mathbf{u}^\star$ and $\mathbf{u}^{\star\star}$. Indeed:

- **Fully implicit discretization**. As the name indicates, this scheme is obtained by considering:

  $$\mathbf{u}^\star = \mathbf{u}^{\star\star} = \mathbf{u}^{n+1}. \tag{30}$$

  It obviously grants the best results in terms of precision and stability but it keeps the non-linearity of the equation (even in a Chorin-Temam method), which should be adressed using a fixed-point or a Newton method.

- **Semi-implicit discretization**. It consists in taking:

$$\mathbf{u}^\star = 2\mathbf{u}^n - \mathbf{u}^{n-1} \qquad \mathbf{u}^{\star\star} = \mathbf{u}^{n+1}. \tag{31}$$

  The advantage of this discretization is the "linearization" of the problem without going into severe problems concerning stability. This is a good tradeoff between precision, simplicity and stability in many problems (not all). This choice is justified by looking at the second order extrapolation of $\mathbf{u}^{n+1}$ as function of $\mathbf{u}^n$ and $\mathbf{u}^{n-1}$, which then is the explicit discretization of the value at time t^{n+1} using the values at the two previous time steps.

$$\begin{aligned} \mathbf{u}(t^{n+1}) &= \zeta \mathbf{u}(t^n) + \eta \mathbf{u}(t^{n-1}) && (32) \\ &= (\zeta + \eta)\, \mathbf{u}(t^{n+1}) + \Delta t\, (-\zeta - 2\eta)\, \partial_t \mathbf{u}(t^{n+1}) + O(\Delta t^2), && (33) \end{aligned}$$

  where the space variable is not listed for the sake of notation. Enforcing consistency up to first order yields:

$$\zeta = 2 \qquad \eta = -1. \tag{34}$$

  This is:

$$\mathbf{u}^\star = 2\mathbf{u}^n - \mathbf{u}^{n-1}. \tag{35}$$

- **Fully explicit discretization**. We take:

$$\mathbf{u}^\star = \mathbf{u}^{\star\star} = 2\mathbf{u}^n - \mathbf{u}^{n-1}. \tag{36}$$

References

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