Modal analysis for the linearized Rayleigh-Bénard problem

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The starting point is the dimensionless form of the problem, which reads:

$$\begin{cases} \frac{1}{P}\left(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla\varpi + R\theta\mathbf{e}_z + \Delta\mathbf{u} \\ \partial_t\theta - u_z + \mathbf{u}\cdot\nabla\theta = \Delta\theta \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{1}$$

where P and R are dimensionless parameters given by:

$$P = \frac{\nu}{\chi} \qquad R = \frac{\alpha g \Delta T h^3}{\nu\chi}, \tag{2}$$

and ϖ is the dimensionless pressure variable along with $\mathbf{e}_z$ the unit vector along z and $u_z := \mathbf{u}\cdot\mathbf{e}_z$. We assume $|\mathbf{u}| \ll 1$ and $\theta \ll 1$ (infinitesimal), so that we can linearize the equations, meanining that the non-linear terms (i.e. $\mathbf{u}\cdot\nabla\mathbf{u}$ and $\mathbf{u}\cdot\nabla\theta$) are supposed to be negligible. This yields:

$$\begin{cases} \frac{1}{P}\partial_t \mathbf{u} = -\nabla\varpi + R\theta\mathbf{e}_z + \Delta\mathbf{u} \\ \partial_t\theta - u_z = \Delta\theta \\ \nabla\cdot\mathbf{u} = 0. \end{cases} \tag{3}$$

We remember that for a vector field $\mathbf{v} = (v_x, v_y, v_z)$ written with respect to the cartesian system (that is not true for cylindrical and other systems of coordinates), the curl is given by:

$$\nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z \tag{4}$$

We take the curl of the first equation of (3), having (we will eventually develop some more terms):

$$\frac{1}{P}\partial_t\left(\nabla\times\mathbf{u}\right) = -\underbrace{\nabla\times\left(\nabla\varpi\right)}_{=0} + R\left(\frac{\partial\theta}{\partial y}\mathbf{e}_y - \frac{\partial\theta}{\partial x}\mathbf{e}_y\right) + \Delta\left(\nabla\times\mathbf{u}\right), \tag{5}$$

where we have used the fact that the curl of the gradient is always vanishing. Taking the curl again for Eq. (5), gives:

$$\frac{1}{P}\partial_t\left(\nabla\times\nabla\times\mathbf{u}\right) = R\left(\frac{\partial^2\theta}{\partial z\partial y}\mathbf{e}_y - \frac{\partial^2\theta}{\partial y^2}\mathbf{e}_z + \frac{\partial^2\theta}{\partial z\partial x}\mathbf{e}_x - \frac{\partial^2\theta}{\partial x^2}\mathbf{e}_z\right) + \Delta\left(\nabla\times\nabla\times\mathbf{u}\right). \tag{6}$$

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We now develop the term $\nabla \times \nabla \times \mathbf{u}$ using the rule for the computation of the curl (twice) that we have seen so far. Instead of writing the whole result, we directly projet it on the z direction:

$$(\nabla \times \nabla \times \mathbf{u}) \cdot \mathbf{e}_z = \frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} - \frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2}. \tag{7}$$

Before plugging back (7) in (6), we use the incompressibility constraint from (3) to simplify the expression. The constraint reads, in cartesian coordinates:

$$0 = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \qquad \rightarrow \qquad \frac{\partial^2 u_y}{\partial y \partial z} = -\frac{\partial^2 u_x}{\partial x \partial z} - \frac{\partial^2 u_z}{\partial z^2}. \tag{8}$$

We are left with:

$$(\nabla \times \nabla \times \mathbf{u}) \cdot \mathbf{e}_z = -\frac{\partial^2 u_z}{\partial x^2} - \frac{\partial^2 u_z}{\partial y^2} - \frac{\partial^2 u_z}{\partial z^2}. \tag{9}$$

As a matter of fact, Eq. (6) projected along z rewrites:

$$-\frac{1}{P}\partial_t \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = -R \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - \Delta \left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right). \tag{10}$$

Figuring out the fact that the Laplacian of a scalar σ in cartesian coordinates is defined as:

$$\Delta \sigma := \frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} + \frac{\partial^2 \sigma}{\partial z^2}, \tag{11}$$

we have:

$$-\frac{1}{P}\partial_t \left(\Delta u_z \right) = -R \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - \Delta \left(\Delta u_z \right). \tag{12}$$

Thus, for the moment, the linearized system we are dealing with reads:

$$\begin{cases} -\frac{1}{P}\partial_t \left(\Delta u_z \right) & = -R \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) - \Delta \left(\Delta u_z \right) \\ \partial_t \theta - u_z & = \Delta \theta. \end{cases} \tag{13}$$

We understand that the z direction is somehow “spacial” in the system, thus we can suppose that for every planar cut of domain with normal vector parallel to $\mathbf{e}_z$, the solution behaves in (x, y) following some Fourier mode with wavenumber k_x in x and k_y in y. We shall note $|\mathbf{k}|^2 := k_x^2 + k_y^2$. Thene, we suppose that:

$$u_z = u_z(t, x, y, z) = e^{\lambda t} e^{i(k_x x + k_y y)} f(z) \tag{14}$$

$$\theta = \theta(t, x, y, z) = e^{\lambda t} e^{i(k_x x + k_y y)} g(z), \tag{15}$$

with $\lambda \in \mathbb{C}$ and f and g real functions to be found, describing the dependence of the solution of the z coordinate. We not plug this ansatz into the system (13). It is advisable to perform the computation term by term:

- $$\partial_t(\Delta u_z) = \partial_t \left(\left(-|\mathbf{k}|^2 f(z) + f''(z) \right) e^{\lambda t} e^{i(k_x x + k_y y)} \right) \tag{16}$$

  $$= \lambda \left(\frac{d^2}{dz^2} - |\mathbf{k}|^2 \right) \left(f(z) \right) e^{\lambda t} e^{i(k_x x + k_y y)}. \tag{17}$$

From now on, formulae like $\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)$ should be understood as linear operators acting on f.

- $$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = -(k_x^2 + k_y^2)e^{\lambda t}e^{i(k_x x + k_y y)}g(z) = -|\mathbf{k}|^2 e^{\lambda t}e^{i(k_x x + k_y y)}g(z). \tag{18}$$

- $$\Delta(\Delta u_z) = \Delta\left(\left(-|\mathbf{k}|^2 f(z) + f''(z)\right)e^{\lambda t}e^{i(k_x x + k_y y)}\right) \tag{19}$$
$$= \left(|\mathbf{k}|^4 - 2|\mathbf{k}|^2\frac{d^2}{dz^2} + \frac{d^4}{dz^4}\right)(f(z)))e^{\lambda t}e^{i(k_x x + k_y y)} \tag{20}$$
$$= \left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)^2 (f(z)))e^{\lambda t}e^{i(k_x x + k_y y)}. \tag{21}$$

- $$\partial_t\theta = \lambda e^{\lambda t}e^{i(k_x x + k_y y)}g(z). \tag{22}$$

- $$\Delta\theta = \left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)(g(z))\,e^{\lambda t}e^{i(k_x x + k_y y)} \tag{23}$$

With all this in mind, we are ready to rewrite the equation from (13) under a new form. Keeping in mind that the term $e^{\lambda t}e^{i(k_x x + k_y y)}$ can be simplified everywhere (almost everywhere to be honest), the first equation becomes:

$$-\frac{\lambda}{P}\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)f(z) = R|\mathbf{k}|^2 g(z) - \left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)^2 f(z), \tag{24}$$

which can be easily rewritten under the form:

$$\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2 - \frac{\lambda}{P}\right)f(z) = R|\mathbf{k}|^2 g(z). \tag{25}$$

We are almost done: the second equation has become:

$$\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2 - \lambda\right)g(z) = -f(z). \tag{26}$$

We want to eliminate $g(z)$ from (25) in order to have an equation only on $f(z)$. By multiplying (25) and using (26), we obtain:

$$\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2 - \lambda\right)\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2\right)\left(\frac{d^2}{dz^2} - |\mathbf{k}|^2 - \frac{\lambda}{P}\right)f(z) = -R|\mathbf{k}|^2 f(z), \tag{27}$$

Quod erat demonstrandum.